Abstract—We show an effective cut-free variant of Glivenko’s theorem extended to formulas with weak quantifiers (those without eigenvariable conditions): “There is an elementary function \( f \) such that if \( \varphi \) is a cut-free \( \text{LK} \) proof of \( \vdash A \) with symbol complexity \( \leq \alpha \), then there exists a cut-free \( \text{LJ} \) proof of \( \vdash \neg\neg A \) with symbol complexity \( \leq f(\alpha) \).” This follows from the more general result: “There is an elementary function \( f \) such that if \( \varphi \) is a cut-free \( \text{LK} \) proof of \( A \vdash A \) with symbol complexity \( \leq \alpha \), then there exists a cut-free \( \text{LJ} \) proof of \( A \vdash \neg\neg A \) with symbol complexity \( \leq f(\alpha) \).” The result is proved using a suitable variant of cut-elimination by resolution (CERES) and subsumption.

I. INTRODUCTION

In mathematics there are theorems whose proofs can be drastically shortened by working in more powerful axiom systems. The first result of this kind has been the Gödel speed-up theorem [1]–[3], an effective variant of the second incompleteness theorem (some natural examples of this phenomenon can be found in [4]). The most general speed-up result in the literature is the theorem of Ehrenfeucht and Mycielski [5]:

If the theory \( T + \neg\alpha \) is undecidable, then there is no recursive function \( f \) such that \( |A|_T \leq f(|A|_{T+\alpha}) \), for every sentence \( A \) provable in \( T \).

Where \( ||S|| \) denotes the symbol complexity of \( S \) and \( |A|_T \) is the proof complexity of \( A \) in \( T \), i.e. the minimal size of a proof of \( A \) in \( T \). Note that, for \( T + \neg\alpha \) being undecidable, it is necessary that \( \alpha \) is not a theorem of \( T \), otherwise \( T + \neg\alpha \) would be inconsistent and therefore decidable.

The Ehrenfeucht-Mycielski theorem is formulated \textit{prima facie} for classical logic systems, but holds for intuitionistic logic systems as well. Consider that \( T \) is intuitionistic logic (\( \text{IL} \)) and let \( \alpha \) be a sentence \( A \) such that \( \text{IL} + \neg\neg A \) is undecidable. Take the sequents \( S_B = A \vdash \neg\neg A \rightarrow B \) for all first-order sentences \( B \). Obviously, there are recursively bound proofs of the \( S_B \) in \( \text{IL} \) and thus recursively bound proofs of the formulas \( \neg\neg A \rightarrow B \) in \( \text{IL} + A \). Assume that there is a function \( f \) such that \( |C|_{\text{IL}} \leq f(|C|_{\text{IL} + A}) \) for every sentence \( C \). Then if we take \( C_B = \neg\neg A \rightarrow B \), it must be the case that the formulas \( C_B \) have recursively bound proofs in \( \text{IL} \). But note that this would mean that the formulas \( B \) have recursively bound proofs in \( \text{IL} + \neg A \), contradicting its undecidability.

The Ehrenfeucht-Mycielski theorem however does not apply to the relation of intuitionistic and classical first-order logic, as intuitionistic logic extended by the negation of any instance of the \textit{ tertium non datur} is in fact inconsistent and therefore decidable. Nevertheless, there are non-recursively bounded speed-ups of classical over intuitionistic logic. Take for example the sentences \( F_B : B \lor \neg B \), where \( B \) ranges over all intuitionistically valid first-order sentences. Clearly all \( F_B \) have bounded classical proofs (linear in \( ||B|| \), the size of \( B \)). On the other hand there exists an infinite sequence \( B_n \) of intuitionistically provable formulas for which there exists no recursive function \( f \) with \( ||B_n||_{\text{LJ}} \leq f(||B_n||) \) for all \( n \) (follows from the undecidability of first-order intuitionistic logic). This gives us the following speed-up result: there exists no recursive function \( f \) such that \( |A|_{\text{LK}} \leq f(|A|_{\text{LJ}}) \) for all formulas \( A \) provable in intuitionistic first-order logic.

Another example of a nonrecursive speed-up by classical over intuitionistic logic is the monadic fragment [6]. In general it is therefore interesting to investigate the general conditions under which speed-ups can be achieved.

On the relation of classical and intuitionistic proofs, the best known result is Glivenko’s theorem, also called double negation translation:

\textit{An arbitrary propositional formula} \( A \) \textit{is classically provable if and only if} \( \neg\neg A \) \textit{is intuitionistically provable}.

In this paper we consider the extension of Glivenko’s double negation translation to formulas \( A \) with weak quantifiers only. Note that Glivenko’s theorem does not hold if strong quantifiers are admitted, e.g., \( \forall x.\neg\neg A(x) \rightarrow \forall x.A(x) \) is classically provable but \( \neg\neg(\forall x.\neg\neg A(x) \rightarrow \forall x.A(x)) \) is not intuitionistically provable in general. There is of course a polynomial double negation translation with respect to the fragment with weak quantifiers of cut-free \( \text{LK} \) proofs into \( \text{LJ} \) proofs, but this translation introduces cuts (see Section IV-A). Therefore, it does not provide an elementary bound with respect to the relation of cut-free proofs.

The main result of this paper is the following: there is an elementary function \( f \) such that if \( \varphi \) is a cut-free \( \text{LK} \) proof of \( A \vdash \text{A} \) with symbol complexity \( \leq \alpha \), then there exists a cut-free \( \text{LJ} \) proof of \( A \vdash \text{A} \) with symbol complexity \( \leq f(\alpha) \). An
elementary bound for the double negation translation follows immediately. As a corollary we obtain: if subclasses of \( \textbf{LK} \) proofs of sequents \( A \vdash A \) admit elementary cut-elimination, the same holds for \( \textbf{LJ} \).

This paper uses variants of CERES to obtain the above mentioned results. CERES [7] is a global semi-semantical cut-elimination method which subsumes the usual stepwise reduction methods and allows a better transformation of cut-free proofs due to its global nature.

II. PRELIMINARIES

We will describe now the exact setting in which we prove our results. We will represent classical proofs in the (multiplicative) calculus \( \textbf{LK} \), depicted in Figure 1. Proofs will be considered intuitionistic if they are in the \( \textbf{LJ} \) calculus, which is obtained by simply restricting \( \textbf{LK} \) to sequents with at most one formula on the right side. Consequently, \( \textbf{LJ} \) has no contraction right rule. We will use the term proof to denote a derivation from tautological axioms (i.e., \( A \vdash A \)) and use derivation otherwise.

Definition 1 ((Negative) sequent). A sequent is a structure \( S : \Gamma \vdash \Delta \) where \( \Gamma \) and \( \Delta \) are multisets of formulas. \( S \) is called negative if \( \Delta \) is empty.

Definition 2 (Polarity). Let \( F \) be a formula and \( F' \) a subformula of \( F \). Then we can define the polarity of \( F' \) in \( F \), i.e., \( F' \) can be positive or negative in \( F \), according to the following criteria:

- If \( F \equiv F' \), then \( F' \) is positive in \( F \).
- If \( F \equiv A \land B \) or \( F \equiv A \lor B \) or \( F \equiv \forall x.A \) or \( F \equiv \exists x.A \) and \( F' \) is positive (negative) in \( A \) or \( B \), then \( F' \) is positive (negative) in \( F \).
- If \( F \equiv A \rightarrow B \) and \( F' \) is positive (negative) in \( B \), then \( F' \) is positive (negative) in \( F \).
- If \( F \equiv A \rightarrow B \) and \( F' \) is positive (negative) in \( A \), then \( F' \) is negative (positive) in \( F \).
- If \( F \equiv \neg A \) and \( F' \) is positive (negative) in \( A \), then \( F' \) is negative (positive) in \( F \).

Definition 3 (Strong and weak quantifiers). Let \( F \) be a formula. If \( \forall x \) occurs positively (negatively) in \( F \), then \( \exists x \) is called \( \forall x \) is called a strong (weak) quantifier. If \( \exists x \) occurs positively (negatively) in \( F \), then \( \forall x \) is called \( \exists x \) is called a weak (strong) quantifier. Let \( A_1, \ldots, A_n \vdash B_1, \ldots, B_m \) be a sequent. A quantifier is called strong (weak) in this sequent if it is strong (weak) in the corresponding formula \( \forall x \ldots \forall x.A_1 \land \ldots \land A_n \rightarrow B_1 \lor \ldots \lor \lor B_m \).

Strong quantifiers in a sequent will be those introduced by the inferences \( \lor \) and \( \exists \).

Definition 4 (Skolemized sequent). A sequent is called skolemized if it contains no strong quantifiers.

In this paper we investigate proofs of skolemized negative sequents. Observe that, in this class of proofs, classical and intuitionistic provability coincide (which would not be the case if strong quantifiers in the end-sequents were allowed). Nevertheless, this does not mean that the classical and intuitionistic proofs of such sequents will be the same. In this paper, we establish a relation between them, namely, that they have about the same complexity. More precisely, we show that within this class there is an elementary transformation of classical cut-free proofs into intuitionistic cut-free proofs. This transformation is based on the cut-elimination method CERES.

III. CUT-ELIMINATION BY NEGATIVE RESOLUTION

This cut-elimination procedure for (a sub-class of) intuitionistic logic is a modification of the CERES method (cut-elimination by resolution) for classical logic [7] and was first described in [8]. It can be split into the following steps:

1. Extraction of the characteristic clause set \( \text{iL}(\varphi) \).
2. Refutation of \( \text{iL}(\varphi) \) using the negative refinement of resolution.
3. Extraction of a set of projections \( \pi(C) \) for every \( C \in \text{iL}(\varphi) \).
4. Merging of refutation and projections into a proof with only atomic cuts.
5. Left-shift cut-elimination of the atomic cuts.

The method uses the concept of formula ancestors, which we define now.

\[
\frac{\Delta \vdash A}{\Gamma, A \vdash A} \quad \frac{\Gamma \vdash A, P \quad \Gamma, P \vdash \Delta}{\Gamma, P, \Delta \vdash A} \quad \frac{\Gamma, P \vdash \Delta, P \quad \Gamma, P \vdash \Delta}{\Gamma, P, \Delta \vdash \Delta} \quad \frac{\Gamma \vdash A}{\Gamma, P, \Delta \vdash A} \quad \frac{\Gamma, P \vdash \Delta}{\Gamma, \neg P \vdash \Delta} \quad \frac{\Gamma \vdash \Delta, P \quad \Gamma \vdash \Delta, \neg P}{\Gamma \vdash \Delta, P \land \neg P}
\]

\[
\frac{P, \Gamma \vdash \Delta}{\Gamma \vdash \Delta \lor P} \quad \frac{\Gamma \vdash \Delta, Q \quad \Gamma \vdash \Delta, P}{\Gamma \vdash \Delta, P \lor Q} \quad \frac{P, \Gamma \vdash \Delta}{\Gamma \vdash \Delta \lor \neg P} \quad \frac{P \lor P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, \neg P} \quad \frac{\Gamma \vdash \Delta, \neg P \lor P}{\Gamma \vdash \Delta}
\]

\[
\frac{P, \forall x. \Gamma, \Delta}{\Gamma, \Delta} \quad \frac{P, \Gamma \vdash \Delta, \exists x. \Gamma}{\Gamma, \Delta \lor \exists x. \Gamma} \quad \frac{P, \forall x. \Gamma \vdash \Delta}{\Gamma, \Delta \lor \exists x. \Gamma} \quad \frac{P, \forall x. \Gamma \vdash \Delta}{\Gamma, \Delta \lor \exists x. \Gamma} \quad \frac{P, \forall x. \Gamma \vdash \Delta}{\Gamma, \Delta \lor \exists x. \Gamma}
\]

\[
\frac{P \lor P, \Delta \vdash \Delta}{\Gamma \vdash \Delta} \quad \frac{P \lor P, \Delta \vdash \Delta}{\Gamma \vdash \Delta} \quad \frac{P \lor P, \Delta \vdash \Delta}{\Gamma \vdash \Delta} \quad \frac{P \lor P, \Delta \vdash \Delta}{\Gamma \vdash \Delta} \quad \frac{P \lor P, \Delta \vdash \Delta}{\Gamma \vdash \Delta}
\]

\[
\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta} \quad \frac{\Gamma, P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, P} \quad \frac{\Gamma, P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, P} \quad \frac{\Gamma, P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, P} \quad \frac{\Gamma, P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, P}
\]

\[
\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta} \quad \frac{\Gamma, P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, P} \quad \frac{\Gamma, P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, P} \quad \frac{\Gamma, P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, P} \quad \frac{\Gamma, P, \Delta \vdash \Delta}{\Gamma \vdash \Delta, P}
\]

Figure 1. \( \textbf{LK} \): Sequent calculus for classical logic. It is assumed that: \( \alpha \) is a variable not contained in \( P, \Gamma \) or \( \Delta; t \) does not contain variables bound in \( P; i \in \{1, 2\} \); and \( A \) is an atomic formula.
Definition 5 (Formula ancestor). Let $\nu$ be a formula occurrence in a sequent calculus proof $\varphi$. If $\nu$ is a principal formula occurrence of an inference then the occurrences of the auxiliary formula (formulas) in the premises are ancestors of $\nu$. If $\nu$ is principal formula of a weakening or occurs in an axiom then $\nu$ has no ancestor. If $\nu$ is not a principal occurrence then the corresponding occurrences in contexts of the (premise) premises are ancestors of $\nu$. The ancestor relation is then defined as the reflexive transitive closure.

We will use the following proof as our running example to clarify each definition:

\[
\begin{align*}
A \vdash A & \quad \neg_l \quad A \vdash A & \quad \neg_l \\
\neg A, A \vdash \neg A & \quad \neg_l & & \neg A, A \vdash A & \quad \neg_l & & A, B, \neg A & \quad \text{cut} \\
\neg A, A \vdash B & \quad r & & C \vdash C & \quad \text{cut} & & C \vdash C & \quad \text{cut} \\
\neg A \vdash A & \quad r & & \neg A \vdash A & \quad r & & C \vdash C & \quad \text{cut} & & \neg A, C \vdash (A \rightarrow B) \land C
\end{align*}
\]

The clause set extraction consists in, intuitively, collecting all atomic ancestors of the cut formulas which occur in the axioms of the proof. The clauses are formed depending on how these atoms are related via binary inferences in the proof.

Definition 6 (Clause). A sequent $\Gamma \vdash \Delta$ is called a clause if $\Gamma$ and $\Delta$ are multisets of atoms.

Definition 7 (Characteristic clause-set). Let $\varphi$ be a proof of a skolemized sequent. The characteristic clause set is built recursively from the leaves of the proof until the end sequent. Let $\nu$ be a sequent in this proof. Then:

- If $\nu$ is an axiom, then $\text{CL}(\nu)$ contains the sub-sequent of $\nu$ composed only of cut ancestors.
- If $\nu$ is the result of the application of a unary rule on a sequent $\mu$, then $\text{CL}(\nu) = \text{CL}(\mu)$
- If $\nu$ is the result of the application of a binary rule on sequents $\mu_1$ and $\mu_2$, then we distinguish two cases:
  - If the rule is applied to ancestors of the cut formula, then $\text{CL}(\nu) = \text{CL}(\mu_1) \cup \text{CL}(\mu_2)$
  - If the rule is applied to ancestors of the end sequent, then $\text{CL}(\nu) = \text{CL}(\mu_1) \times \text{CL}(\mu_2)$

Where\(^1\): $\text{CL}(\mu_1) \times \text{CL}(\mu_2) = \{C \circ D | C \in \text{CL}(\mu_1), D \in \text{CL}(\mu_2)\}$

If $\nu_0$ is the root node $\text{CL}(\nu_0)$ is called the characteristic clause set of $\varphi$.

The clause set of our example proof contains one negative clause:

$$\text{CL}(\varphi) = \{A \vdash C; A, C \vdash; \vdash A, C; C \vdash A\}$$

The next step is to obtain a negative resolution refutation of $\text{CL}(\varphi)$. It is thus important to show that this set is always refutable.

Theorem 1. Let $\varphi$ be a proof of a skolemized end-sequent. Then the characteristic clause set $\text{CL}(\varphi)$ is refutable.

Proof: In [7], [9].

Definition 8 (Resolution calculus). The resolution calculus consists of the following rules:

$$\Gamma \vdash \Delta, A; \Gamma' \vdash \Delta' \quad R \quad \Gamma, A, A' \vdash \Delta \quad C_l \quad \Gamma \vdash \Delta, A, A' \quad C_r$$

Where $\sigma$ is the most general unifier of $A$ and $A'$.

Definition 9 (Negative resolution refinement). A resolution derivation is called negative if, in every application of the rule $R$, one of the clauses in the premise is negative and the only factoring rule is $C_l$ applied to negative clauses, i.e. all rules are of the form:

$$\Gamma \vdash \Delta, A; \Gamma' \vdash \Delta' \quad R \quad \Gamma, A, A' \vdash \Delta \quad C_l$$

Theorem 2. The negative resolution refinement is complete.

Proof: By Theorem 3.6.1. in [10] and by sign renaming.

We give a negative resolution refutation of $\text{CL}(\varphi)$:

\[
\begin{align*}
C \vdash A & \quad A, C \vdash \quad R & & A \vdash C & \quad A, C \vdash \quad R \\
C \vdash C & \quad c_l & & A \vdash A & \quad c_l & & A \vdash A & \quad R
\end{align*}
\]

From Theorems 1 and 2 we conclude that there is always a negative resolution refutation of the clause set. But in order to use this refutation in our method, it needs to be grounded. To this aim we transform the refutation to tree form, rename the variables of clauses in the leaves and apply the global unifier of all the resolutions; finally all variables can be replaced by a constant symbol. For details see [9].

Since our running example is a propositional proof, the resolution refutation is already grounded.

Each clause in the clause set will have a projection associated with it. A projection of a clause $\Delta$ is a derivation built from $\varphi$ by taking the axioms in which the atoms of $C$ occur and all the inferences that operate on end-sequent ancestors. As a result, the end-sequent of a projection will be the end-sequent of $\varphi$ plus the atoms of $C$.

Definition 10 (Projections). Let $\varphi$ be a proof and $\xi$ the last (lower most) inference with conclusion $\nu$. We define $p(\nu)$ as the set of projections $\{\pi(C)|C \in \text{CL}(\nu)\}$. Each projection $\pi(C)$ is a cut-free proof of the sequent $\nu \circ C$.

- If $\xi$ is an axiom, then $p(\nu) = \{\varphi\}$.
- If $\xi$ is a unary rule with premise $\mu$:
Atomic cuts. 

Refutation, it is possible to build a proof \( \phi \) (where these atoms are marked with the end-sequent are present after constructing the projection, formulas of an inference. Moreover, if not all formulas of \( \phi \) is an \( LJ \) derivation. 

Projections of negative clauses are always valid intuitionistic derivations. 

The only modification of negative CERES over the CERES normal form is: 

\[ \begin{array}{c}
A \vdash A' \\
A \rightarrow B, A' \rightarrow_r \\
C \vdash C' \\
C' \vdash C^*, C \\
\neg A, C \vdash A^*, C^*, (A \rightarrow B) \\
\end{array} \]

\[ \begin{array}{c}
w_r \\
w_r \\
w_r \\
w_1 \\
\end{array} \]

We define \( C \star \phi \) is already defined and thus \( C \star \phi \).

We define \( C \star \phi : \)

\[ \begin{array}{c}
C \star \phi_1 \\
C \star \phi_2 \\
C \circ \phi_1 \\
C \circ \phi_2 \\
C \circ \phi \\
\end{array} \]

\[ \begin{array}{c}
S_1 \\
S_2 \\
C \circ S_1 \\
C \circ S_2 \\
C \circ S \\
\end{array} \]

\[ \begin{array}{c}
C \circ S \\
C \circ S \\
C \circ S \\
\end{array} \]

\[ \begin{array}{c}
C \circ S \\
C \circ S \\
C \circ S \\
\end{array} \]

\[ \begin{array}{c}
C \circ S \\
C \circ S \\
C \circ S \\
\end{array} \]

Note that the formulas in \( C \) will come from both branches, and we are working in a multiplicative calculus, after applying a binary rule we need to contract the formulas from \( C \) to obtain the correct multi-set.

### Definition 12 (Negative CERES normal form).

Let \( \phi \) be an \( LJ \) proof of a negative skolemized sequent \( S \), \( CL(\phi) \) its clause set and \( \phi \) a grounded negative resolution refutation of \( CL(\phi) \). We first construct \( \phi' = S \star \phi \). Note that this is a derivation of \( S \) from a set of axioms \( C \circ S \), with \( C \in CL(\phi) \), which are exactly the end-sequents of the projections \( \pi(C) \) of \( \phi \). Now we define \( \phi(\phi) \) by replacing all axioms of \( \phi' \) by the respective projections. By definition, \( \phi(\phi) \) is an \( LJ \) proof of \( S \) with only atomic cuts. We call it the negative CERES normal form of \( \phi \) with respect to \( \phi \).

This procedure of obtaining a negative CERES normal form from an \( LJ \) proof \( \phi \) is called negative CERES and we will denote the proof with atomic cuts obtained by \( \phi \). The only modification of negative CERES over the CERES method is the enforcement of negative resolution.

Since we are using negative resolution and the end-sequent of \( \phi \) is negative, every atomic cut in \( \phi \) will have the shape:
\[
\frac{A \vdash A}{\Delta, A \vdash} \text{cut}
\]

\[
\frac{\Gamma, A \vdash}{\Gamma, \Delta \vdash A, \Gamma' \vdash} \text{cut}
\]

Figure 3. Elimination of the cut in left-shift cut-elimination.

Note that, since projections might be classical proofs, \(\hat{\phi}\) may also be a classical proof. Nevertheless it can be transformed again into an intuitionistic proof by removing the atomic cuts.

Since the original proof was in a single conclusion calculus, we know that every sequent with more than one formula on the right side must contain at most one end-sequent ancestor, the other formulas being atomic cut-ancestors. Therefore, if we can eliminate the atomic cuts maintaining always at most one end-sequent ancestor on the right side of every sequent, we will obtain an \(\mathbf{LJ}\) proof. Now we show how to achieve this by insisting on a specific discipline for reductive cut-elimination.

Definition 13 (Left-shift cut-elimination). Let \(\hat{\phi}\) be a proof with only atomic cuts. We call left-shift cut-elimination the process of removing the atomic cuts that, starting from the top most cuts down, (1) permutes the cut over all the rules of its left branch (Figure 2) until reaching an axiom and (2) eliminates the cut by using the proof on its right branch (Figure 3).

Theorem 4. Let \(\phi\) be an \(\mathbf{LJ}\) proof with cuts and \(\hat{\phi}\) the negative CERES normal form obtained with negative CERES. Then eliminating the cuts from \(\hat{\phi}\) using left-shift cut-elimination yields an \(\mathbf{LJ}\) proof.

Proof: Although \(\phi\) is an \(\mathbf{LJ}\) proof, each inference \(\rho\) in \(\hat{\phi}\) might be applied to a multiple conclusion sequent because of atomic cut-ancestors. By reductively eliminating the cuts, we make sure that the resulting proof’s sequents contain no atomic cut-ancestors on the right, but there is no guarantee that they will all be single conclusion. This can be ensured by two things: (1) \(\phi\) is a proof of a negative sequent and (2) left-shift cut-elimination is used to eliminate the atomic cuts from \(\hat{\phi}\).

Let \(\rho\) be an inference in \(\hat{\phi}\) that was an instance of an inference in \(\phi\) (which was originally applied to a single conclusion sequent). We have thus to show that after left-shift cut-elimination, every \(\rho\) will be applied to a single conclusion sequent.

First note that every inference \(\rho\) is applied to a sequent such that its right context contain at most one end-sequent ancestor, the other formulas being atomic cut-ancestors. Now observe that in the reduction rules of Figure 2, the \(\rho\) in the resulting derivation is always applied to a sequent whose right context contains strictly less formulas then in the original derivation. Moreover, these are all the rules necessary for eliminating the atomic cuts, as there is no right contraction of the cut-formulas because there is no right contraction in the negative resolution fragment. After eliminating all the cuts, every \(\rho\) will be applied to a sequent whose right context contains at most one end-sequent ancestor and no cut ancestors, exactly as it was in \(\phi\).

Second, upon actually eliminating the cut (Figure 3), the derivation used is a negative projection which, by Theorem 3, is an \(\mathbf{LJ}\) proof.

The final proof is therefore a valid \(\mathbf{LJ}\) proof.

IV. TRANSFORMING CLASSICAL INTO INTUITIONISTIC PROOFS

A. The double negation translation for the weak quantifier fragment

As (quantified) cuts always contain quantifiers in both polarities, we have to translate cut-free \(\mathbf{LK}\) derivations.

Definition 14 (Left-sided intuitionistic translation). Let \(\phi\) be a cut-free \(\mathbf{LK}\) proof containing only weak quantifiers with end-sequent \(\Gamma \vdash \Delta\). We define the left-sided intuitionistic translation \(I(\phi)\) an \(\mathbf{LJ}\) proof with cuts and end-sequent \(\neg \neg \Gamma, \neg \neg \Delta \vdash\) (where \(\neg \neg \Gamma\) and \(\neg \Delta\) denote the sets \(\{\neg F | F \in \Gamma\}\) and \(\{\neg F | F \in \Delta\}\), respectively) inductively on \(\phi\):

- If \(\phi\) consists of only one axiom \(C \vdash\), then \(I(\phi)\) is the \(\mathbf{LJ}\) derivation:

\[
\frac{C \vdash}{\neg \neg C \vdash \neg I} \quad \frac{\neg C \vdash}{\neg \neg C \vdash \neg r} \quad \frac{\neg \neg C \vdash}{\neg C \vdash \neg l}
\]

- If \(\phi\) ends with a structural rule, then we can assume, by induction, that \(I(\phi')\) is already defined and \(I(\phi)\) is straightforward:

\[
\begin{align*}
\frac{\Gamma, C \vdash \Delta}{\neg \neg \Gamma, \neg \neg C, \neg \neg \Delta \vdash} & \quad \frac{I(\phi')}{\neg \neg \Gamma, \neg \neg C, \neg \neg \Delta \vdash} & \quad \text{cut}_l \\
\frac{\Gamma, C \vdash \Delta}{\neg \neg \Gamma, \neg \neg C, \neg \neg \Delta \vdash} & \quad \frac{I(\phi')}{\neg \neg \Gamma, \neg \neg C, \neg \neg \Delta \vdash} & \quad \text{cut}_r \\
\frac{\Gamma \vdash \Delta}{\neg \neg \Gamma, \neg \neg \Delta \vdash} & \quad \frac{I(\phi')}{\neg \neg \Gamma, \neg \neg \Delta \vdash} & \quad \text{cut}_u_l \\
\frac{\Gamma, C \vdash \Delta}{\neg \neg \Gamma, \neg \neg C, \neg \neg \Delta \vdash} & \quad \frac{I(\phi')}{\neg \neg \Gamma, \neg \neg C, \neg \neg \Delta \vdash} & \quad \text{cut}_u_r
\end{align*}
\]
If \( \varphi \) ends with a logical rule, we distinguish the cases in Figure 4 (in all cases we assume, by induction, that \( I(\varphi') \) and \( I(\varphi'') \) are defined).

**Definition 15.** Let \( \varphi \) be an \( \text{LK} \) or \( \text{LJ} \) proof. We define the size of \( \varphi \), denoted by \( \|\varphi\| \), as the number of symbol occurrences in the proof.

**Proposition 1.** Let \( \varphi_K \) be a cut-free \( \text{LK} \) derivation of \( \Gamma \vdash B \). Then it can be translated into an \( \text{LJ} \) derivation with cuts of \( \neg\Gamma \vdash \neg\neg B \), denoted by \( \varphi_I \), such that \( \|\varphi_I\| \leq \|\varphi_K\|^2 \).

*Proof:* The left-sided intuitionistic translation of \( \varphi_K \), gives us an intuitionistic proof \( I(\varphi_K) \) with end-sequent \( \neg\neg\Gamma, \neg B \vdash \). By simply applying a \( \neg\neg \) to the end-sequent of this proof, we obtain an \( \text{LJ} \) derivation with cuts of \( \neg\neg\Gamma \vdash \neg\neg B \). The number of inserted proofs in \( I(\varphi_K) \) (indicated by \( \forall \)) is linear in the number of inferences in \( \varphi_K \) and so is their size.

Note that not much is known about the complexity of the elimination of intuitionistic cuts with quantifiers of the same polarity and arbitrary propositional structure. Therefore the proof transformation given in Proposition 1 does not give us an elementary bound on the mapping of cut-free \( \text{LK} \) to cut-free \( \text{LJ} \) proofs within this class of proofs.

**B. An elementary translation of cut-free proofs**

**Definition 16.** Let \( \varrho \) be a resolution refutation. We define the size of \( \varrho \), denoted by \( \|\varrho\| \), as the number of symbol occurrences in \( \varrho \).

**Theorem 5.** There exists an elementary function \( h \) s.t., given a proof \( \varphi \), a resolution refutation \( \varrho \) of \( \text{CL}(\varphi) \) and a corresponding \( \text{CERES} \) normal form \( \hat{\varphi} \), we have \( \|\hat{\varphi}\| \leq h(\|\varphi\|, \|\varrho\|) \).

*Proof:* In [9] Lemma 6.5.3 it is shown that there is a strictly increasing elementary function \( H \) s.t., given a proof \( \varphi \) and a resolution refutation \( \varrho \) of \( \text{CL}(\varphi) \), \( \|\varrho\| \leq H(\|\varphi\|, l(\varrho)) \) for an appropriate ground resolution refutation \( \varrho' \) of \( \text{CL}(\varphi) \) (where \( l(\varrho) \) is the number of node proofs in \( \varrho \)). As \( l(\varrho) \leq \|\varrho\| \) we get

\[
(1) \quad \|\varrho'\| \leq H(\|\varphi\|, \|\varrho\|).
\]

By Lemma 6.5.4 in [9] we have

\[
(2) \quad \|\hat{\varphi}\| \leq H(\|\varphi\|, l(\varrho)) + \|\varrho\| * r(\varrho')
\]

for the \( \text{CERES} \) normal form \( \hat{\varphi} \) corresponding to \( \varrho \), where \( r(\varrho') \) is the size of a maximal term occurring in \( \varrho' \). Clearly \( r(\varrho') \leq \|\varrho'\| \). Therefore, putting (1) and (2) together we obtain

\[
\|\hat{\varphi}\| \leq H(\|\varphi\|, \|\varrho\|) + H(\|\varphi\|, \|\varrho\|) = H(\|\varphi\|, \|\varrho\|).
\]

In defining \( h(x, y) = H(x, y)^2 * x \) (\( h \) is clearly elementary as \( H \) is) we eventually obtain

\[
\|\hat{\varphi}\| \leq h(\|\varphi\|, \|\varrho\|).
\]

**Theorem 6.** Let \( \hat{\varphi} \) be a negative \( \text{CERES} \) normal form of \( \varphi \) and let \( \varrho_0 \) be the cut-free proof obtained after applying left-shift cut-elimination to \( \hat{\varphi} \). Then there exists a constant \( c \) (independent of \( \varphi \)) such that \( \|\varrho_0\| \leq c \|\hat{\varphi}\| \).

*Proof:* Given the transformations in Figures 2 and 3 (which are all the rules necessary for eliminating the atomic cuts), observe that the right-hand side uses only those derivations that were already present on the left, without duplicates. Thus left-shift cut-elimination does not increase the number of inference nodes in the proof. As the transformation in Figure 3 even eliminates an inference, \( \varphi_0 \) contains less inferences than \( \hat{\varphi} \), provided there is at least one cut in \( \hat{\varphi} \). Still the rules in Figure 2 may mildly increase the symbolic size of a proof. Note that e.g. in the first rule we may have \( \|\varphi^*\| > \|\varphi\| \) (\( \rho \) may be \( \forall \) and a large term is eliminated (top-down) by the rule which now occurs twice in the result). But this increase happens for every rule

\[
\begin{array}{c}
\Xi_1 \vdash \Delta^*, A \quad \Xi_2 \vdash \Delta, A \quad \Xi_3 \vdash \Delta, A \quad \Xi_4 \vdash \Delta, A \\
\Gamma, \Gamma^* \vdash \Delta^*, A \quad \Gamma, \Gamma^* \vdash \Delta^*, A \quad \Gamma, \Gamma'' \vdash \Delta^*, A \quad \Gamma, \Gamma'' \vdash \Delta^*, A
\end{array}
\]

\[
\begin{array}{c}
\Xi_1 \vdash \Delta^*, A \quad \Xi_2 \vdash \Delta^*, A \quad \Xi_3 \vdash \Delta^*, A \quad \Xi_4 \vdash \Delta^*, A \\
\Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A
\end{array}
\]

\[
\begin{array}{c}
\Xi_1 \vdash \Delta^*, A \quad \Xi_2 \vdash \Delta^*, A \quad \Xi_3 \vdash \Delta^*, A \quad \Xi_4 \vdash \Delta^*, A \\
\Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A
\end{array}
\]

\[
\begin{array}{c}
\Xi_1 \vdash \Delta^*, A \quad \Xi_2 \vdash \Delta^*, A \quad \Xi_3 \vdash \Delta^*, A \quad \Xi_4 \vdash \Delta^*, A \\
\Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A
\end{array}
\]

\[
\begin{array}{c}
\Xi_1 \vdash \Delta^*, A \quad \Xi_2 \vdash \Delta^*, A \quad \Xi_3 \vdash \Delta^*, A \quad \Xi_4 \vdash \Delta^*, A \\
\Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A
\end{array}
\]

\[
\begin{array}{c}
\Xi_1 \vdash \Delta^*, A \quad \Xi_2 \vdash \Delta^*, A \quad \Xi_3 \vdash \Delta^*, A \quad \Xi_4 \vdash \Delta^*, A \\
\Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A
\end{array}
\]

\[
\begin{array}{c}
\Xi_1 \vdash \Delta^*, A \quad \Xi_2 \vdash \Delta^*, A \quad \Xi_3 \vdash \Delta^*, A \quad \Xi_4 \vdash \Delta^*, A \\
\Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A \quad \Gamma, \Gamma'' \vdash \Delta, A
\end{array}
\]
The transformation $I(\varphi)$ for logical connectives. In all the derivations, $\forall$ indicate an easy intuitionistic proof.

Figure 4.
ρ (coming from a left-hand-side of a cut) only once, and the material causing the increase is already present in the original proof. Therefore, there exists a constant c such that: \( \|\varphi_0\| \leq c \|\hat{\varphi}\| \).

\[ \text{Proof:} \] Let us consider a cut-free (classical) proof \( \varphi \) of the skolemized sequent \( S: A_1, \ldots, A_n \vdash \) (i.e. there are no strong quantifiers in the \( A_i \) and - in classical logic - the \( A_i \) are equivalent to \( \forall\)-prenex forms). Now consider our transformation \( T(\varphi) \):

\[
\begin{align*}
A_1, \ldots, A_n \vdash & A \vdash \neg: \Gamma \vdash r \\
& \vdash \neg A, A_1, \ldots, A_n \vdash \text{cut}
\end{align*}
\]

where \( A = A_1 \land \cdots \land A_n \) and \( \psi \) is a cut-free intuitionistic proof of length polynomial in \( \|\varphi\| \). \( T(\varphi) \) proves the same end-sequent as \( \varphi \). Now observe that the cut-formula \( \neg A \) on the left branch of the cut has only weak quantifiers, and only strong quantifiers on the right branch. Therefore, in classical logic, the cut-formula is equivalent to a \( \Sigma_1 \)-formula and the cut has the strength of a \( \Sigma_1 \)-cut only. Even without this detour over prenexing, it is clear that the cut-elimination on cut \( \neg A \) is elementary (in fact it is at most double exponential) by just following the Gentzen reduction of the quantifiers. Moreover it will suffice to eliminate the quantifiers in the cuts only. So let \( \chi \) be the proof \( T(\varphi) \) after elimination of the quantifiers in the cuts (so \( \chi \) is a proof with quantifier-free cuts only). Putting things together there exists an elementary function \( h \) s.t. \( \|\chi\| \leq h(\|T(\varphi)\|) \); so there exists also an elementary function \( f \) with \( \|\chi\| \leq f(\|\varphi\|) \).

Now negative CERES comes into play: consider the characteristic clause set \( \text{CL}(T(\varphi)) \) and let \( C' \) be the characteristic clause set of \( \chi \). Note that \( C' \) is a set of ground clauses (indeed we may assume that in a proof containing no strong quantifiers only ground terms are introduced by the quantifier rules). As \( \chi \) is elementary in \( T(\varphi) \), \( C' \) is as well by Proposition 2. By Proposition 3, the main subsumption result about reductive cut-elimination, \( \text{CL}(T(\varphi)) \) subsumes \( C' \) and every resolution refutation \( \rho' \) of \( C' \) is subsumed by a resolution refutation \( \rho \) of \( \text{CL}(T(\varphi)) \). As \( C' \) is ground, the length of a shortest negative resolution refutation \( \rho' \) is at most exponential in \( \|C'\| \) (note that the number of different negative clauses definable over the ground atoms is at most exponential). Moreover, \( \|\rho\| \leq \|\rho'\| \) (and \( \rho \) is also a negative resolution refutation as negative clauses only be subsumed by negative clauses or by the empty clause). Clearly \( \rho' \) is elementary in \( T(\varphi) \) and so is \( \rho \), i.e. \( \|\rho\| \leq g(\|T(\varphi)\|) \) for an appropriate elementary function \( g \) (independent of \( \rho \) and \( \varphi \)). So we refute \( \text{CL}(T(\varphi)) \) with \( \rho \) and get a CERES-normal form \( \hat{\varphi} \). By Theorem 5 we obtain \( \|\hat{\varphi}\| \leq h(\|T(\varphi)\|, \|\rho\|) \) and thus \( \|\varphi\| \leq h(\|T(\varphi)\|, g(\|T(\varphi)\|)) \).

Define \( h' \) as \( h'(x) = h(x, g(x)) \); then \( h' \) is elementary and \( \|\hat{\varphi}\| \leq h'(\|T(\varphi)\|) \).

As \( \rho' \) is the shortest negative resolution refutation of of \( C' \) there are no tautological clauses occurring in \( \rho' \) (note that a shortest negative resolution refutation never contains
As a consequence also \( \phi \) does not contain tautological clauses. Now consider the proof \( T(\varphi) \). As all inferences in \( \varphi \) (within \( T(\varphi) \)) go into the cut formula \( \neg A \), the clauses of the characteristic clause sets coming from \( \varphi \) are all tautologies. But these tautologies are not used in \( \varphi \).

It follows that all projections in the CERES normal form of \( T(\varphi) \), denoted here by \( \hat{\varphi} \), come from the intuitionistic part of the proof. But note that, in this case, \( \hat{\varphi} \) can be transformed into an intuitionistic cut-free proof \( \psi \) via the method described in Theorem 4. By Theorem 6 also this transformation is elementary and \( ||\psi|| \leq g(||\varphi||) \), where \( g \) is an appropriate elementary function put together by the bound functions above.

We illustrate the transformation of Theorem 7 with an example. Let \( \varphi \) be the \( \text{LK} \) proof:

\[
\begin{align*}
\frac{Pf_\alpha \vdash Pf_\alpha}{Pa \vdash Pf_\alpha, Pf_j \vdash} & \quad \frac{Pf_\alpha \vdash}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} \\
\frac{P_\alpha \vdash}{Pf_\alpha \vdash} & \quad \frac{Pf_\alpha \vdash}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} \\
\frac{\exists x.(Px \rightarrow Pfx) \exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\exists x.(Px \rightarrow Pfx) \vdash}{\vdash} \\
\frac{\neg \exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\neg \exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\neg \exists x.(Px \rightarrow Pfx) \vdash}{\vdash}
\end{align*}
\]

Then we can construct \( \Xi = T(\varphi) \), which proves the same end-sequent but has a full intuitionistic proof on the right branch of the cut:

\[
\begin{align*}
\frac{P_\alpha \vdash}{Pf_\alpha \vdash} & \quad \frac{Pf_\alpha \vdash}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} \\
\frac{Pf_\alpha \vdash}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} & \quad \frac{Pf_\alpha \vdash}{P_\alpha \vdash} \\
\frac{\exists x.(Px \rightarrow Pfx) \exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\exists x.(Px \rightarrow Pfx) \vdash}{\vdash} \\
\neg \exists x.(Px \rightarrow Pfx) \vdash & \quad \neg \exists x.(Px \rightarrow Pfx) \vdash & \quad \neg \exists x.(Px \rightarrow Pfx) \vdash
\end{align*}
\]

We apply the negative CERES method to this proof. The clause set extracted is the following:

\[
\text{CL}(\Xi) = \{ Pf_\alpha \vdash Pf_\alpha ; \vdash P_\alpha ; Pf_\alpha \vdash \}
\]

Note that the tautological clause \( Pf_\alpha \vdash Pf_\alpha \), which came from the classical part of \( \Xi \) can be eliminated. The only possible (negative) refutation is \( \varphi \):

\[
\vdash P_\alpha \vdash Pf_\alpha \vdash a \quad Pf_\alpha \vdash \alpha \leftarrow a
\]

Since \( \varphi \) uses clauses that come from the intuitionistic side of \( \Xi \), these are the only projections we need:

\[
\pi(\vdash P_\alpha) :
\]

\[
\pi(Pf_\alpha \vdash) :
\]

Note that the projection of the negative clause is intuitionistic, but the other one is classical. Then we can compute the CERES normal form \( \hat{\varphi} \):

\[
\begin{align*}
\frac{Pf_\alpha \vdash Pf_\alpha}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} \\
\frac{Pf_\alpha \vdash}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} \\
\frac{P_\alpha \vdash}{Pf_\alpha \vdash} & \quad \frac{Pf_\alpha \vdash}{P_\alpha \vdash} \\
\frac{\exists x.(Px \rightarrow Pfx) \exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\exists x.(Px \rightarrow Pfx) \vdash}{\vdash}
\end{align*}
\]

By performing left-shift cut-elimination, we obtain the \( \text{LJ} \) proof \( \psi \):

\[
\begin{align*}
\frac{Pf_\alpha \vdash Pf_\alpha}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} \\
\frac{Pf_\alpha \vdash}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} \\
\frac{Pf_\alpha \vdash}{P_\alpha \vdash} & \quad \frac{P_\alpha \vdash}{Pf_\alpha \vdash} \\
\frac{\exists x.(Px \rightarrow Pfx) \exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\exists x.(Px \rightarrow Pfx) \vdash}{\vdash} & \quad \frac{\exists x.(Px \rightarrow Pfx) \vdash}{\vdash}
\end{align*}
\]

The complexity of this transformation, according to the description of Theorem 7, has an exponential tower of three. We believe nevertheless that this bound is not sharp and that a double exponential bound exists, though not easy to prove.

**Corollary 1.** There exists an elementary function \( h \) with the following property: given a classical cut-free proof \( \varphi \)
of \( \vdash A \) (where \( A \) is a formula without strong quantifiers) there exists a cut-free intuitionistic proof \( \psi \) of \( \vdash \lnot \lnot A \) s.t. \( \|\psi\| \leq h(\|\varphi\|) \).

Proof: We extend \( \varphi \) by a \( \lnot : l \) rule and obtain a cut-free proof \( \varphi' \) of \( \lnot A \vdash \). By Theorem 7 there exists an intuitionistic cut-free proof \( \varphi' \) of \( \lnot A \vdash l \) and \( \|\varphi'\| \leq g(\|\varphi\|) \). We obtain an intuitionistic cut-free proof of \( \lnot \lnot A \) just by appending \( \lnot: r \) to \( \varphi' \). An elementary function \( h \) with \( \|\psi\| \leq h(\|\varphi\|) \) can be constructed from \( g \) in an obvious way.

V. CONCLUSION

In this paper we describe an elementary transformation of cut-free \( \mathbf{LK} \) proofs of skolemized sequents of the form \( S: \Gamma \vdash \) into cut-free \( \mathbf{LJ} \) proofs of \( S \). As a corollary, we get an elementary bound for Glivenko’s double negation translation on sequents with weak quantifiers.

Concerning other double negation translations, we believe that, by using a suitable Skolemization of intuitionistic logic, the results of this paper may be extended to Kuroda translations. However, the Gödel-Gentzen-Kolmogoroff translation might not admit elementary bounds on translations of cut-free classical to cut-free intuitionistic proofs. A reason can be found, e.g., in the translation of Figure 5. We recall that cuts can be encoded as tautologies via \( \lnot \rightarrow \) inferences, therefore any proof with cuts can be linearly translated to a cut-free proof of the same end-sequent extended by tautologies. By using the translation in Figure 5, the tautologies are removed, and this is equivalent to removing the cuts. Thus the transformation must be of non-elementary complexity. Of course, the calculi \( \mathbf{LK} \) and \( \mathbf{LJ} \) should be extended by the axiom \( \vdash \top \). On the other hand, if the top clause of the case distinction is deleted the bound is linear.

It would be interesting to know more about the embeddings of logics on the proof theoretical level. The main result of this paper can be considered as a first step into the direction of comparing translations from analytic (in this case cut-free) to analytic proofs. The crucial tool for achieving the main result of this paper is the method CERES, which, as a global method of proof analysis (compared to the local reductive ones), also allows a more global comparison of proofs.

\[
\begin{align*}
\Psi(\lnot A) & \equiv \top \text{ if } \lnot A \text{ is a substitution instance of a tautology } \\
\lnot \lnot \Psi(A) \text{ otherwise }, \\
\Psi(A \circ B) & \equiv \top \text{ if } A \circ B \text{ is a substitution instance of a tautology } \\
\lnot \lnot (\Psi(A) \circ \Psi(B)) \text{ otherwise, where } \circ \in \{\land, \lor, \rightarrow\}, \\
\Psi(Qx.A(x)) & \equiv \lnot \lnot Qx.\Psi(A(x)) \text{ for } Q \in \{\forall, \exists\}, \\
\Psi(A) & \equiv \lnot \lnot A \text{ if } A \text{ is atomic.}
\end{align*}
\]

Figure 5.

REFERENCES


